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Feedback Nash equilibria in the scalar infinite horizon LQ-game

by
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Abstract: In this note we consider linear stationary feedback Nash equilibria of the scalar linear-quadratic differential game. The planning horizon considered is assumed to be infinite. We present both necessary and sufficient conditions on the system parameters for existence of a unique solution of the associated algebraic Riccati equations (ARE) that stabilizes the closed-loop system. For the case of more solutions, singleton-valued refinements of the equilibrium concept are studied.

Keywords: Linear quadratic games, feedback Nash equilibrium, solvability conditions, Riccati equations

I. Problem statement

In this note we consider the problem where two parties (henceforth called players) try to minimize their individual quadratic performance criterion. Each player controls a different set of inputs to a single system. The system is described by the following differential equation

$$\dot{x} = ax + b_1u_1 + b_2u_2, \quad x(0) = x_0. \quad (1)$$

Here x is the state of the system, u_i is a (control) variable player i can manipulate, x_0 is the arbitrarily chosen initial state of the system, a, b_1 , and b_2 are constant system parameters, and \dot{x} denotes the time derivative of x . The performance criterion player $i = 1, 2$ aims to minimize is:

$$J_i(u_1, u_2) := \frac{1}{2} \int_0^\infty \{x(t)^T q_i x(t) + u_i(t)^T r_{ii} u_i(t)\} dt.$$

We assume that both q_i and r_{ii} are positive and b_i differs from zero.

We consider in detail the existence of a feedback Nash equilibrium of this differential game (see Başar and Olsder (1995) for a precise definition of this equilibrium concept). Closely related to this problem are the following set of coupled algebraic Riccati equations (ARE):

$$-ak_1 - k_1a - q_1 + k_1s_1k_1 + k_1s_2k_2 + k_2s_2k_1 = 0; \quad (2)$$

$$-ak_2 - k_2a - q_2 + k_2s_2k_2 + k_2s_1k_1 + k_1s_1k_2 = 0; \quad (3)$$

where $s_i := b_i r_{ii}^{-1} b_i$.

From Başar and Olsder (1995, proposition 6.8), we have:

Theorem 1: Let $\bar{k}_i \geq 0$ solve the set of Riccati equations. Then the pair of strategies given by

$$u_i = -r_{ii}^{-1} b_i \bar{k}_i x \quad (4)$$

$i = 1, 2$, provide a feedback Nash equilibrium, leading to the cost $J_i(u_1, u_2) := x_0 \bar{k}_i x_0$, for player i .

Moreover, the resulting system dynamics described by $\dot{x} = a_{cl}x$; $x(0) = x_0$, with $a_{cl} := a - s_1 \bar{k}_1 - s_2 \bar{k}_2$, is asymptotically stable. \square

In fact, one can immediately deduce from Weeren (1995, p.96) that when the players are restricted at the outset to memoryless strategies (cf. Lukes (1971)) then existence of a positive solution to the above scalar Riccati equations is a both necessary and sufficient condition for existence of a feedback Nash equilibrium.

Weeren *et al.* (1999) showed that the above set of algebraic Riccati equations (ARE) either has one or three positive solutions. In this paper we study the conditions on the system parameters under which these different situations occur. Furthermore, in case there are three equilibria, we compare these regarding stability and aggregate efficiency in order to reach a singleton-valued refinement of this equilibrium concept.

II. Characterization of the number of equilibria

To study the intersection points of the (ARE) equations we introduce (for notational convenience) the variables:

$$\sigma_i := s_i q_i \text{ and } \kappa_i := s_i k_i, \quad i = 1, 2.$$

Using this notation (2,3) can be rewritten as

$$-2a\kappa_1 - \sigma_1 + \kappa_1^2 + 2\kappa_1\kappa_2 = 0; \quad (5)$$

$$-2a\kappa_2 - \sigma_2 + \kappa_2^2 + 2\kappa_2\kappa_1 = 0. \quad (6)$$

The above question can therefore be reformulated as under which conditions the two hyperbola

$$\kappa_2 = a - \frac{1}{2}\kappa_1 + \frac{\sigma_1}{2} \frac{1}{\kappa_1}; \quad (7)$$

and

$$\kappa_1 = a - \frac{1}{2}\kappa_2 + \frac{\sigma_2}{2} \frac{1}{\kappa_2}; \quad (8)$$

intersect in the first quadrant of the (κ_1, κ_2) -plane.

To visualize the situation, we sketched in figure 1 the hyperbola (7) and (8), respectively.

From (5) we have that $\kappa_2 = \frac{2a\kappa_1 + \sigma_1 - \kappa_1^2}{2\kappa_1}$. Substitution of this into (6) shows that $\kappa_1 > 0$ must satisfy the following equation

$$3\kappa_1^4 - 8a\kappa_1^3 + (4a^2 - 2\sigma_1 + 4\sigma_2)\kappa_1^2 - \sigma_1^2 = 0. \quad (9)$$

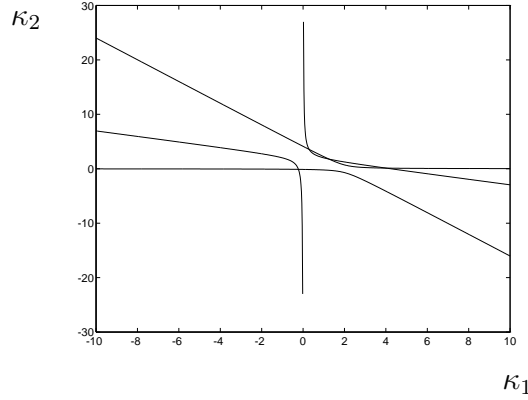


Figure 1: The hyperbola in case $a = 2, \sigma_1 = 1, \sigma_2 = \frac{1}{2}$

From this equation it is immediately clear that the set of (ARE) equations has at most four solutions. A more detailed analysis of this equation shows that

Theorem 2: (ARE) has exact 1 solution in the first quadrant if and only if the system parameters satisfy either one of the following two conditions:

$$a \leq 0 \quad (10)$$

$$3((a^2 + \sigma_1 + \sigma_2)^2 - 3\sigma_1\sigma_2)^2 - 4a^2(a^2 + \sigma_1 + \sigma_2)^3 > 0. \quad (11)$$

In all other cases, (ARE) has three solutions in the first quadrant. \square

Proof: Consider the function

$$f(x) := 3x^4 - 8ax^3 + (4a^2 - 2\sigma_1 + 4\sigma_2)x^2 - \sigma_1^2. \quad (12)$$

Differentiation of f yields

$$f'(x) := 12x^3 - 24ax^2 + 2(4a^2 - 2\sigma_1 + 4\sigma_2)x.$$

So, the stationary points of f are

I) if $a^2 - 2\sigma_2 + \sigma_1 < 0$: $x_1 = 0$

II) if $a^2 - 2\sigma_2 + \sigma_1 \geq 0$: $x_1 = 0$; $x_2 = a + \frac{1}{3}\sqrt{3}\sqrt{a^2 - 2\sigma_2 + \sigma_1}$ and $x_3 = a - \frac{1}{3}\sqrt{3}\sqrt{a^2 - 2\sigma_2 + \sigma_1}$.

In case I) it is clear that f can only have one positive zero. In case II) things are somewhat more complicated. To analyze this case, first consider the situation that $x_3 \leq 0$ or, equivalently, either $a \leq 0$ or $2a^2 + 2\sigma_2 - \sigma_1 \leq 0$ and $a > 0$. Elementary analysis shows then that f has one positive zero again. In case $x_3 > 0$ it is easily verified that f has exact one positive zero if and only if the product $f(x_2)f(x_3) > 0$. Elementary calculations show that $\frac{27}{16}f(x_2)f(x_3) = -a^8 + 6a^4(\sigma_2^2 - \sigma_1\sigma_2 + \sigma_1^2) + 3(\sigma_2^2 - \sigma_1\sigma_2 + \sigma_1^2)^2 + 4a^2(\sigma_1 - 2\sigma_2)(\sigma_1 + \sigma_2)(2\sigma_1 - \sigma_2)$ (i). Consequently $f(x_2)f(x_3) > 0$ if and only if (11) holds. So, (ARE) has exactly one solution if either 1) $a \leq 0$; 2) $a^2 - 2\sigma_2 + \sigma_1 \leq 0$; 3) $2a^2 + 2\sigma_2 - \sigma_1 \leq 0$ or 4) (11) holds. Next we show that if either condition 2) or 3) is satisfied condition 4) also holds. To that end we rewrite σ_2 as $\sigma_2 = \frac{\sigma_1 + a^2}{2} + \mu$ for some positive μ . Substitution of this expression into (i) yields $27\{\frac{1}{4}(a^4 - \sigma_1^2) + a^2\mu\}^2 + \frac{9}{2}\mu^2(\sigma_1^2 - a^4) + 14a^2\mu^3 + 3\mu^4$. Now, if $a^4 - \sigma_1^2 \geq 0$, then $27\{\frac{1}{4}(a^4 - \sigma_1^2) + a^2\mu\}^2 - \frac{9}{2}\mu^2a^4 \geq 27a^4\mu^2 - \frac{9}{2}\mu^2a^4 \geq 0$. So, (i) is positive. On the other hand, if $a^4 - \sigma_1^2 < 0$, it is obvious that (i) is positive too, which proves the claim. In a similar way one can show that 4) holds in case 3) is satisfied by rewriting σ_1 as $\sigma_1 = 2a^2 + 2\sigma_2 + \mu$, for some positive μ . \square

To get an impression how the parameter surface satisfying (11) (for $a > 0$) looks like we use the reparametrization $\sigma_i = a^2\tau_i$. For a fixed a we visualized this set, $3((1 + \tau_1 + \tau_2)^2 - 3\tau_1\tau_2)^2 - 4(1 + \tau_1 + \tau_2)^3 > 0$, in figure 2.

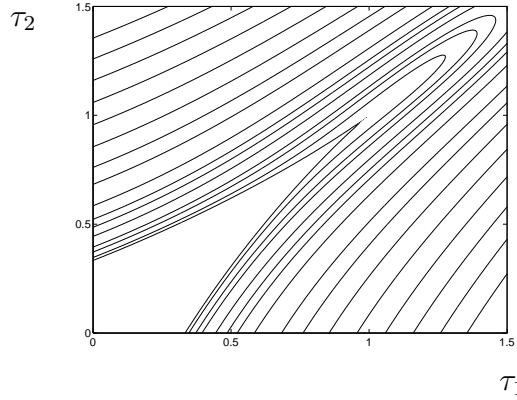


Figure 2: \equiv : Set where $3((1 + \tau_1 + \tau_2)^2 - 3\tau_1\tau_2)^2 - 4(1 + \tau_1 + \tau_2)^3 \geq 0$.

We conclude this section by stressing the fact

Corollary 3: If the system parameters do not satisfy any of the conditions mentioned in theorem 2, the differential game has at least three feedback Nash solutions. \square

III. Stability of solutions

A natural question that arises in case none of the conditions of theorem 2 is satisfied is whether it is possible to formulate some (natural) additional requirement on the feedback Nash solution that is satisfied by only one of the solutions. In Weeren et al. (1999) this issue was also raised, and it is shown that the additional requirement of dynamic stability of the solutions is not a sufficient criterion to eliminate this nonuniqueness. If there are three equilibria, see Figure 2, the smallest and largest equilibrium (in terms of κ_1) are stable equilibria whereas the remaining equilibrium is a saddle-point. An interesting point, that is not elaborated in the above mentioned paper but which is easily verified, is that in the symmetric case, $\sigma_1 = \sigma_2$, the origin $(0,0)$ is located on a saddle-path. This suggests that, on the one hand, one can expect computational difficulties in finding the appropriate equilibrium in a finite planning horizon setting with no penalties on the final state. On the other hand, one can conclude that the equilibrium outcomes of the game will depend in that case crucially on the assumptions how to penalize outcomes of the final state of the game.

Next, we consider the closed-loop stability of the solutions as a criterion to eliminate nonuniqueness. Our first result is

Theorem 4: Assume that $\sigma_1 \neq \sigma_2$.

Then the closed-loop "matrix" a_{cl} differs for all solutions.

Moreover, in case (ARE) has three different solutions (k_1, k_2) , (l_1, l_2) and (m_1, m_2) , with (without loss of generality) $k_1 < l_1 < m_1$, then the smallest closed-loop matrix is attained by either (k_1, k_2) or (m_1, m_2) .

Proof: From theorem 1 we have that if (k_1, k_2) solve (ARE), the corresponding closed-loop matrix a_{cl} equals $a - s_1 k_1 - s_2 k_2$ or, equivalently, $a - \kappa_1 - \kappa_2$. Using this we note that (5,6) can be rewritten as

$$2a_{cl}\kappa_1 + \sigma_1 + \kappa_1^2 = 0; \quad (13)$$

$$2a_{cl}\kappa_2 + \sigma_2 + \kappa_2^2 = 0. \quad (14)$$

Now assume, both the two different solutions (k_1, k_2) and (l_1, l_2) satisfy the above equation yielding the same closed-loop matrix a_{cl} . From (13) we then get:

$$2a_{cl}k_1 + \sigma_1 + k_1^2 = 0 \text{ and } 2a_{cl}l_1 + \sigma_1 + l_1^2 = 0.$$

Subtracting these two equations yields: $2a_{cl}(k_1 - l_1) + (k_1^2 - l_1^2) = 0$. Since $k_1 \neq l_1$ we conclude that $2a_{cl} = -(k_1 + l_1)$. Substituting this into (13) gives $-(k_1 + l_1)k_1 = -\sigma_1 - k_1^2$ or, equivalently, $k_1l_1 = \sigma_1$. Similarly, one can show that under these assumptions also the equality $k_2l_2 = \sigma_2$ must hold. Furthermore, by assumption $a_{cl} = a - k_1 - k_2 = a - l_1 - l_2$. So, $k_1 + k_2 = l_1 + l_2$. Substitution of the above derived equalities $k_i l_i = \sigma_i$ into this equation yields then

$$\frac{\sigma_1}{k_1} + \frac{\sigma_2}{k_2} = k_1 + k_2. \quad (15)$$

On the other hand we have from (13,14) that

$$-2a_{cl} = \frac{\sigma_1}{k_1} + k_1 = \frac{\sigma_2}{k_2} + k_2. \quad (16)$$

Addition of (15) and (16) yields $k_1k_2 = \sigma_2$ (i), whereas subtraction of them gives rise to the equality $k_1k_2 = \sigma_1$ (ii). Comparing (i) and (ii) we conclude that the equality $\sigma_1 = \sigma_2$ must hold, which violates our assumption. So, our assumption that both solutions yield the same closed-loop system must be wrong, which completes the first part of the proof.

To prove the second part of the theorem, we first note that all three solutions lie on the hyperbola (7). Note that on the interval $[k_1, m_1]$ this hyperbola is a convex linesegment. It is easily seen that the maximum of the function $g(\kappa_1, \kappa_2)$ defined by $g(\kappa_1, \kappa_2) := \kappa_1 + \kappa_2$ on this convex linesegment is attained at either one of the endpoints of the segment. Since both closed-loop matrices do not coincide (as we proved above), this shows that the minimum value of $a_{cl} = a - \kappa_1 - \kappa_2$ is attained at either one of the points (k_1, k_2) or (m_1, m_2) . \square

Next, consider the symmetric case, i.e. $\sigma_1 = \sigma_2$. Obviously, when this condition holds then, whenever (k_1, k_2) satisfies (ARE) also (k_2, k_1) solves these equations (see e.g. (13,14)). In particular the symmetric solution (k, k) is obtained for $k = \frac{a + \sqrt{a^2 + 3\sigma}}{3}$. With respect to the corresponding closed-loop systems we have the following

Theorem 5: Let $\sigma_1 = \sigma_2$. Assume that (ARE) has three different solutions.

Then the closed-loop systems of both non-symmetric solutions coincide. Moreover, the closed-loop matrix a_{cl} of the non-symmetric solution is smaller than that of the symmetric solution.

Proof: First we consider the closed-loop system of the non-symmetric solutions.

Subtracting (14) from (13) yields the equality $2a_{cl}(\kappa_1 - \kappa_2) = -(\kappa_1^2 - \kappa_2^2)$. Since we assume that $\kappa_1 \neq \kappa_2$, we obtain from this equality that $2a_{cl} = -(\kappa_1 + \kappa_2)$. Using the definition of $a_{cl} = a - \kappa_1 - \kappa_2$, we conclude that $2a = \kappa_1 + \kappa_2$. Substitution of this into a_{cl} shows that $a_{cl} = -a$ (note that $a > 0$ since we assumed that (ARE) has three different solutions!). Note that this result does not depend on the specific structure of the asymmetric solution. So, we conclude that the closed-loop matrix for both asymmetric solutions equals $-a$.

To prove the second statement of the theorem, we calculate the closed-loop matrix of the system resulting from the symmetric solution (k, k) , which we will denote by $a_{cl}(k, k)$. Simple calculation shows that $a_{cl}(k, k) = \frac{1}{3}(a - 2\sqrt{a^2 + 3\sigma})$. Since by assumption (ARE) has three different solutions, it follows immediately from theorem 2 that $a^2 > \sigma$. Using this, elementary calculations show then that $a_{cl}(k_1, k_2) < a_{cl}(k, k)$. Which completes the proof. \square

From theorems 4 and 5 we conclude that closed-loop stability is also not a sufficient criterium to eliminate nonuniqueness completely. In the symmetric case ($\sigma_1 = \sigma_2$) things go wrong. On the other hand, we observe that always either at the smallest or largest equilibrium (in terms of κ_1) the most stable closed-loop system is achieved.

Finally, we consider minimum of total cost incurred by both players in the equilibrium as a criterion to eliminate non-uniqueness. To that end we first note that since the cost incurred by a player is given by $J_i = \kappa_i x_0^2$, $i = 1, 2$. Since these equilibrium points lie on one hyperbola, it is clear that never one of these equilibria will be Pareto efficient (that is both players have lower cost in an equilibrium compared to those in another equilibrium). We first prove the next lemma

Lemma 6: Both (\bar{k}_1, \bar{k}_2) and (\bar{k}_2, \bar{k}_1) are equilibria of the game if and only if $\sigma_1 = \sigma_2$.

Proof: Assume that both points are equilibria of the game. Then from (5) we have $-2a\bar{k}_1 - \sigma_1 + \bar{k}_1^2 + 2\bar{k}_1\bar{k}_2 = 0$ and from (6) we have $-2a\bar{k}_1 - \sigma_2 + \bar{k}_1^2 + 2\bar{k}_1\bar{k}_2 = 0$. Subtracting both equations yields then $\sigma_1 = \sigma_2$. The other implication was already noted before. \square

Now, assume $\sigma_1 \neq \sigma_2$. Subtracting (5) from (6) yields

$$-2a(\kappa_1 - \kappa_2) - \sigma_1 + \sigma_2 + (\kappa_1 - \kappa_2)(\kappa_1 + \kappa_2) = 0.$$

or, stated differently,

$$\kappa_1 + \kappa_2 = \frac{\sigma_1 - \sigma_2}{\kappa_1 - \kappa_2} + 2a. \quad (17)$$

If both (k_1, k_2) and (\bar{k}_1, \bar{k}_2) are two different equilibria yielding the same total cost, i.e. $(k_1 + k_2)x_0^2 = (\bar{k}_1 + \bar{k}_2)x_0^2$ (i), it follows then straightforwardly from (17) that also $k_1 - k_2 = \bar{k}_1 - \bar{k}_2$ (ii) should hold. Obviously (i) and (ii) together imply that $k_1 = \bar{k}_1$ and $k_2 = \bar{k}_2$, which contradicts the assumption that both equilibria differ. In case $\sigma_1 = \sigma_2$, it is clear from the proof of theorem 5 that total cost in the smallest and largest equilibrium point are the same, i.e. $2ax_0^2$. Furthermore elementary calculation shows that in the symmetric equilibrium (k, k) total cost are $\frac{2}{3}(a + \sqrt{a^2 + 3\sigma})x_0^2$. Since $\sigma < a^2$ (see theorem 2), total cost are in this equilibrium point always smaller than $2ax_0^2$. This yields

Theorem 7: If $\sigma_1 \neq \sigma_2$ then total cost differ in all equilibria.

If $\sigma_1 = \sigma_2$ then total cost are minimized in the symmetric equilibrium point. \square

So we conclude that the additional requirement, that amongst all equilibria we look for an equilibrium which minimizes total cost, always gives rise to a unique equilibrium.

IV. Concluding remarks

In this paper we studied feedback Nash equilibria in the two-player linear quadratic scalar differential game. We showed that the corresponding set of algebraic Riccati equations has either one or three different solutions in the

first quadrant and we gave necessary and sufficient conditions on the system parameters for existence of a unique solution of these equations.

In order to be able to discriminate between different equilibria we considered three singleton-valued refinements of this equilibrium concept: dynamic stability, stability of the closed-loop system and aggregate efficiency.

We noted that always the smallest and largest solution (measured w.r.t. κ_1) are dynamically stable and the other equilibrium is a saddle point. Furthermore we pointed out that if $\sigma_1 = \sigma_2$ the origin is situated on a saddle path. With respect to closed-loop stabilization we saw that either the largest or smallest solution will stabilize the system most. In case $\sigma_1 = \sigma_2$ closed-stability at the largest and smallest equilibrium point is the same. So in that case, using this criterion, we still have non-uniqueness. If one uses the additional requirement to choose that equilibrium for which the sum of the cost of both players is minimal we showed that always a unique equilibrium results. In particular we saw that if $\sigma_1 = \sigma_2$ the minimal cost are attained in the symmetric equilibrium. However, as we already noted, this equilibrium point is dynamically unstable and its closed-loop stability is worse than in the other equilibria. So, definite answers on which equilibrium one should choose seem to be difficult in that situation.

The obtained results may be helpful in analyzing problems in the area of environmental economics and macro-economic policy coordination problems where this framework is a very natural one to model problems (see e.g. Engwerda (1998) for references). Furthermore, we hope that the obtained results may be helpful in analyzing the more general multi-player and multi-dimensional case.

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